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On Nonnil-Noetherian Rings

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ABSTRACT

Let R be a commutative ring with 1 such that $Nil(R)$ is a divided prime ideal of R. The purpose of this paper is to introduce a new class of rings that is closely related to the class of Noetherian rings. A ring R is called a *Nonnil-Noetherian ring* if every nonnil ideal of R is finitely generated. We show that many of the properties of Noetherian rings are also true for Nonnil-Noetherian rings; we use the idealization construction to give examples of Nonnil-Noetherian rings that are not Noetherian rings; we show that for each $n \geq 1$, there is a Nonnil-Noetherian ring with Krull dimension n which is not a Noetherian ring.

Key Words: Noetherian rings; Finitely generated ideals; Divided ideals; Phi-rings.

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1. INTRODUCTION

We assume throughout that all rings are commutative with $1 \neq 0$. Let R be a ring. Then $T(R)$ denotes the total quotient ring of R, $Nil(R)$ denotes the set of nilpotent elements of R, $Z(R)$ denotes the set of zero-divisor elements of R and $dim(R)$ denotes the Krull dimension of R. Recall from Dobbs (1976) and Badawi (1999b) that a prime ideal of R is called a *divided prime* if $P \subset (x)$ for every $x \in R \setminus P$. In Badawi (1999a, 2000, 2001) the author paid attention to the class of rings $\mathscr{H} = \{R : R$ is a commutative ring and $Nil(R)$ is a divided prime ideal$\}$. A generalization of pseudo-valuation domains that was introduced by Hedstrom and Houston (1978) and a generalization of valuation domains (chained rings) to the context of rings in the class $\mathscr{H}$ were established in Badawi (1999a, 2000, 2001). In this article, we give a generalization of Noetherian (commutative) rings to the context of rings that are in the class $\mathscr{H}$. An ideal I of a ring R is said to be *a nonnil ideal* if $I \not\subset Nil(R)$. Let $R \in \mathscr{H}$. We say that R is a *Nonnil-Noetherian ring* if each nonnil ideal of R is finitely generated. Recall from Badawi (1999a) that for a ring $R \in \mathscr{H}$ with total quotient ring $T(R)$, let $\phi : T(R) \longrightarrow K := R_{Ni_l(R)}$ such that $\phi(a/b) = a/b$ for every $a \in R$ and $b \in R \setminus Z(R)$. Then ϕ is a ring homomorphism from $T(R)$ into K, and ϕ restricted to R is also a ring homomorphism from R into K given by $\phi(x) = x/1$ for every $x \in R$. We say that R is a *Nonnil-Noetherian ϕ-ring* if each nonnil ideal of $\phi(R)$ is a finitely generated ideal of $\phi(R)$.

In the first section of this paper, we show that many of the properties of Noetherian rings are also true for Nonnil-Noetherian rings. In the second section, we use the idealization construction as in Huckaba (1988, Chapter VI) to establish examples of Nonnil-Noetherian rings that are not Noetherian rings; we show that for each $n \geq 1$, there is a Nonnil-Noetherian ring with Krull dimension n which is not a Noetherian ring.

2. BASIC PROPERTIES OF NONNIL-NOETHERIAN RINGS

Throughout this section, $\mathscr{H} = \{R : R$ is a commutative ring and $Nil(R)$ is a divided prime ideal$\}$. For a ring $R \in \mathscr{H}$ with total quotient ring $T(R)$, we define $\phi : T(R) \longrightarrow K := R_{Nil(R)}$ such that $\phi(a/b) = a/b$ for every $a \in R$ and every $b \in R \setminus Z(R)$. Then ϕ is a ring homomorphism from $T(R)$ into K, and ϕ restricted to R is also a ring homomorphism fom R into K given by $\phi(x) = x/1$ for every $x \in R$.

We start this section with the following lemma.

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Lemma 2.1. *Let* $R \in \mathcal{H}$. *Then* $R/Nil(R)$ *is ring-isomorphic to* $\phi(R)/Nil(\phi(R))$.

Proof. Let $\alpha : R \longrightarrow \phi(R)$ such that $\alpha(a) = \phi(a) + Nil(\phi(R))$ for every $a \in R$. It is clear that α is a ring-homomorphism from R ONTO $\phi(R)/Nil(\phi(R))$. Now, $Ker(\alpha) = Nil(R)$. Hence, $R/Nil(R)$ is ring-isomorphic to $\phi(R)/Nil(\phi(R))$. □

Theorem 2.2. *Let* $R \in \mathcal{H}$. *Then R is a Nonnil-Noetherian ring if and only if* $R/Nil(R)$ *is a Noetherian domain.*

Proof. Suppose that R is a Nonnil-Noetherian ring. By Kaplansky (1974, Theorem 8), it suffices to show that every nonzero prime ideal of $D = R/Nil(R)$ is finitely generated. Hence, let Q be a nonzero prime ideal of $D = R/Nil(R)$. Then $Q = P/Nil(R)$ for some nonnil prime ideal P of R. Since P is finitely generated, it is clear that $Q = P/Nil(R)$ is a finitely generated ideal of D. Thus, D is a Noetherian domain. Conversely, suppose that $D = R/Nil(R)$ is a Noetherian domain. Let I be a nonnil ideal of R. Since $Nil(R)$ is a divided ideal, $Nil(R) \subset I$. Hence, $J = I/Nil(R)$ is a finitely generated ideal of D. Thus, say, $J = (i_1 + Nil(R), \ldots, i_n + Nil(R))$ for some i_m's in I. Let x be a nonnilpotent element of I. Then $x + Nil(R) = c_1 i_1 + \cdots + c_n i_n + Nil(R)$ in D for some c_m's in R. Hence, there is a $w \in Nil(R)$ such that $x + w = c_1 i_1 + \cdots + c_n i_n$ in R. Since $x \in I \setminus Nil(R)$, $x \mid w$ in R. Thus, $w = xf$ for some $f \in Nil(R)$. Hence, $x + w = x + xf = x(1 + f) = c_1 i_1 + \cdots + c_n i_n$ in R. Since $f \in Nil(R)$, $1 + f$ is a unit of R. Thus, $x \in (i_1, \ldots, i_n)$. Hence, I is a finitely generated ideal of R. Thus, R is a Nonnil-Noetherian ring. □

It is well-known (Kaplansky, 1974, Theorem 8) that if every prime ideal in a ring R is finitely generated, then R is Noetherian. In light of Theorem 2.2, we have the following similar result.

Corollary 2.3. *Let* $R \in \mathcal{H}$. *If every nonnil prime ideal of R is finitely generated, then R is a Nonnil-Noetherian ring.*

Proof. Suppose that every nonnil prime ideal of R is finitely generated. Then every (nonzero) prime ideal of $D = R/Nil(R)$ is finitely generated. Hence, D is a Noetherian domain by Kaplansky (1974, Theorem 8). Thus, R is a Nonnil-Noetherian ring by Theorem 2.2. □

In light of Lemma 2.1 and Theorem 2.2 we have the following result.

Theorem 2.4. *Let $R \in \mathcal{H}$. The following statements are equivalent:*

1. *R is a Nonnil-Noetherian ring.*
2. *$R/Nil(R)$ is a Noetherian domain.*
3. *$\phi(R)/Nil(\phi(R))$ is a Noetherian domain.*
4. *$\phi(R)$ is a Nonnil-Noetherian ring.*

Proof. **(1)** $\Longrightarrow$ **(2)**. It is clear by Theorem 2.2. **(2)** $\Longrightarrow$ **(3)**. It is clear by Lemma 2.1. **(3)** $\Longrightarrow$ **(4)**. Since $\phi(R) \in \mathcal{H}$, the claim is clear by Theorem 2.2. **(4)** $\Longrightarrow$ **(1)**. Since $\phi(R) \in \mathcal{H}$ is a Nonnil-Noetherian ring, $\phi(R)/Nil(\phi(R))$ is a Noetherian domain by Theorem 2.2. Hence, $R/Nil(R)$ is a Noetherian domain by Lemma 2.1. Thus, R is a Nonnil-Noetherian ring by Theorem 2.2. □

In view of Theorem 2.4. We have the following result.

Corollary 2.5. *Let $R \in \mathcal{H}$. Then R is a Nonnil-Noetherian ring if and only if R is a Nonnil-Noetherian ϕ-ring.*

Proof. Since R is a Nonnil-Noetherian ring iff $\phi(R)$ is a Nonnil-Noetherian ring by Theorem 2.4, the claim is now clear. □

It is shown in Gilmer et al. (1999, Theorem 1.17) that if R is a reduced ring (i.e., $Nil(R) = \{0\}$) and each prime ideal of R has a power that is finitely generated, then R is a Noetherian ring. For a ring $R \in H$, we have the following result.

Theorem 2.6. *Let $R \in \mathcal{H}$. Suppose that each nonnil prime ideal of R has a power that is finitely generated. Then R is a Nonnil-Noetherian ring.*

Proof. Let $D = R/Nil(R)$. Then D is an integral domain and hence a reduced ring. Since every prime ideal of D has the form $P/Nil(R)$ for some prime ideal P of R, we conclude that each prime ideal of D has a power that is finitely generated. Thus, D is Noetherian by Gilmer et al. (1999, Theorem 1.17). Hence, R is a Nonnil-Noetherian ring by Theorem 2.2. □

Theorem 2.7 *Let $R \in \mathcal{H}$. Suppose that R is a Nonnil-Noetherian ring. Then any localization of R is a Nonnil-Noetherian ring, and any localization of $\phi(R)$ is a Nonnil-Noetherian ring.*

Proof. First, observe that any localization of R is an element of $\mathcal{H}$. Let S be a multiplicative subset of R, and suppose that J is a nonnil ideal of R_S.

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Then $J = I_S$ for some nonnil ideal I of R. Since I is finitely generated, we conclude that $J = I_S$ is finitely generated. Now, since $\phi(R)$ is a Nonnil-Noetherian ring by Theorem 2.4, by an argument similar to the one just given, we conclude that any localization of of $\phi(R)$ is a Nonnil-Noetherian ring. □

It is known that (Gilmer, 1992, problem 6, page 370) if R is Noetherian of finite Krull dimension n, then each overring of R has Krull dimension at most n. For a ring $R \in \mathcal{H}$, we have the following.

Theorem 2.8. *Let $R \in \mathcal{H}$ be a Nonnil-Noetherian ring of finite Krull dimension n. Then each overring of R has Krull dimension at most n.*

Proof. Let $D = R/Nil(R)$. Then D is a Noetherian domain by Theorem 3.4. It is clear that $dim(D) = n$. Now, Let S be an overring of R. Since $Nil(R)$ is a divided prime ideal of R, we have $Nil(S) = Nil(R)$ is a prime ideal of S. Thus, $S/Nil(R)$ is an overring of $R/Nil(R)$. Hence, S has Krull dimension at most n by Gilmer (1992, problem 6, page 370). Hence, S has Krull dimension at most n. □

It is known (Kaplansky, 1974, problem 1, page 52) that if R satisfies the ascending chain condition on finitely generated ideals, then R is Noetherian. We have the following similar result.

Theorem 2.9. *Let $R \in \mathcal{H}$. Suppose that R satisfies the ascending chain condition on the nonnil finitely generated ideals. Then R is a Nonnil-Noetherian ring.*

Proof. Let $D = R/Nil(R)$. Then D satisfies the ascending chain condition on the finitely generated ideals. Thus, D is a Noetherian domain by Kaplansky (1974, problem 1, page 6). Hence, R is a Nonnil-Noetherian ring by Theorem 3.4. □

It is known (Kaplansky, 1974, Theorem 144) that if $P \subset Q$ are prime ideals in a Noetherian ring such that there exists a prime ideal properly between them, then there are infinitely many. For a ring $R \in H$ we have the following.

Theorem 2.10. *Let $R \in \mathcal{H}$ be a Nonnil-Noetherian ring, and suppose that $P \subset Q$ are prime ideals in R such that there exists a prime ideal properly between them. Then there are infinitely many.*

Proof. Let $D = R/Nil(R)$. Then D is a Noetherian domain by Theorem 2.2. Suppose that $P \subset Q$ are prime ideals in R such that there exists a prime ideal F properly between them. Then the prime ideal $F/Nil(R)$ of D is properly between the prime ideals $P/Nil(R) \subset Q/Nil(R)$ of D. Hence, there are infinitely many prime ideals of D between $P/Nil(R) \subset Q/Nil(R)$ by Kaplansky (1974, Theorem 144). Thus, there are infinitely many prime ideals of R between $P \subset Q$. □

Let $R \in \mathscr{H}$. Recall from Badawi (2001) that R is said to be a ϕ-*chained ring* if for every $x \in R_{Nil(R)} \backslash \phi(R)$, we have $x^{-1} \in \phi(R)$; equivalently, if for every $a, b \in R \setminus Nil(R)$, either $a \mid b$ in R or $b \mid a$ in R. It is known (Badawi, 2001) that a ϕ-chained ring is quasilocal. The following result is a generalization of Gilmer (1992, Theorem 17.5(2)].

Theorem 2.11. *Let $R \in \mathscr{H}$ be a ϕ-chained ring with maximal ideal $M \neq Nil(R)$. Then R is a Nonnil-Noetherian ring if and only if R has Krull dimension* 1 *and M is a principal ideal of R.*

Proof. Let $D = R/Nil(R)$. Since R is a ϕ-chained ring, it is easy to see that D is a valuation domain. Now, suppose that R is a Nonnil-Noetherian ring. Then D is a Noetherian domain by Theorem 2.2. Since D is a Noetherian domain and a valuation domain, D has Krull dimension 1 and $N = M/Nil(R)$ the maximal ideal of D is a principal ideal by Gilmer (1992, Theorem 17.5(2)). Thus, R has Krull dimension 1. Since $N = M/Nil(R)$ is a principal ideal of D, $N = (m + Nil(R))$ for some $m \in M \setminus Nil(R)$. We will show that $M = (m)$. Let $x \in M \setminus Nil(R)$. Then $x + Nil(R) = mc + Nil(R)$ in D for some $c \in R$. Hence, $x - mc = w \in Nil(R)$. Since $Nil(R)$ is divided in R, we conclude that $x \mid w$ in R. Hence, $w = xf$ for some $f \in Nil(R)$. Thus, $x - xf = mc$. Hence, $x(1 - f) = mc$. Since $f \in Nil(R)$, $1 + f$ is a unit of R. Thus, $m \mid x$ in R. Hence, $x \in (m)$. Thus, $M = (m)$. Conversely, suppose that R has Krull dimension 1 and M is a principal ideal of R. Then D has Krull dimention 1 and $M/Nil(R)$ the maximal ideal of R is a principal ideal of D. Thus, D is a Noetherian domain by Gilmer (1992, Theorem 17.5(2)). Hence, R is a Nonnil-Noetherian ring by Theorem 2.2. □

3. EXAMPLES OF NONNIL-NOETHERIAN RINGS

In this section, we show that for each $n \geq 1$, there is a Nonnil-Noetherian ring with Krull dimension n which is not a Noetherian ring.

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Once again, $\mathscr{H} = \{R : R$ is a commutative ring and $Nil(R)$ is a divided prime ideal$\}$. Our non-domain examples of Nonnil-Noetherian rings are provided by the idealization construction $R(+)B$ arising from a ring R and an R-module B as in Huckaba (1988, Chapter VI). We recall this construction. For a ring R, let B be an R-module. Consider $R(+)B = \{(r, b) : r \in R$, and $b \in B\}$, and let (r, b) and (s, c) be two elements of $R(+)B$. Define:

1. $(r, b) = (s, c)$ if $r = s$ and $b = c$.
2. $(r, b) + (s, c) = (r + s, b + c)$.
3. $(r, b)(s, c) = (rs, bs + rc)$.

Under these definitions $R(+)B$ becomes a commutative ring with identity. We recall the following proposition.

Proposition 3.1 (Huckaba, 1988, Theorem 25.1). *Let R be a ring, B be an R-module. Then The ideal J of R(+)B is prime if and only if $J = P(+)B$ where P is a prime ideal of R. Hence, $dim(R) = dim(R(+)B)$.*

We start with the following lemma.

Lemma 3.2. *Let R be an integral domain, B be an R-module, and $D = R(+)B$. Then $Nil(D)$ is a finitely generated ideal of D if and only if B is a finitely generated R-module.*

Proof. It is clear that $Nil(D) = \{(0, b) : b \in B\}$. Hence, suppose that $Nil(D)$ is a finitely generated ideal of D. Then $Nil(D) = ((0, b_1), \ldots, (0, b_n))$. Now, let $b \in B$. Then $(0, b) = (a_1, c_1)(0, b_1) + \cdots + (a_n, c_n)(0, b_n)$ for some $(a_1, c_1), \ldots, (a_n, c_n) \in D$. Thus, $b = a_1 b_1 + \cdots + a_n b_n$. Hence, B is a finitely generated R-module. Conversely, suppose that B is a finitely generated R-module, say, $B = (b_1, b_2, \ldots, b_n)$. Then it is easy to check that $Nil(D) = ((0, b_1), (0, b_2), \ldots, (0, b_n))$. Hence, $Nil(D)$ is a finitely generated ideal of D. □

Recall from Kaplansky (1974) that an integral domain R with quotient field K is called a G-domain if K is a finitely generated ring over R. We recall the following result.

Proposition 3.3 (Kaplansky, 1974, Theorem 146). *A Noetherian domain R which is not a field is a G-domain if and only if $dim(R) = 1$ and R has only a finite number of maximal ideals.*

In the following theorem, we show that there is a Nonnil-Noetherian ring with Krull dimension 1 that is not a Noetherian ring.

Theorem 3.4. *Let R be a Noetherian domain with quotient field K such that $dim(R) = 1$ and R has infinitely many maximal ideals. Then $D = R(+)K \in \mathcal{H}$ is a Nonnil-Noetherian ring with Krull dimension* 1 *which is not a Noetrherian ring. In particular, $Z(+)Q$ is a Nonnil-Noetherian ring with Krull dimension* 1 *which is not a Noetherian ring* (*where Z is the set of all integer numbers with quotient field Q*).

Proof. By Proposition 3.1, we have $dim(D) = 1$. Since K is not finitely generated ring over R by Proposition 3.3, we conclude that K is not a finitely generated R-module. Hence, $Nil(D)$ is not a finitely generated ideal of D by Lemma 3.2. Thus, D is not a Noetherian ring. By Proposition 3.1, we have $Nil(D) = \{0\}(+)K$ is a prime ideal of D. To show that $Nil(D)$ is divided: let $(0,k) \in Nil(D)$, and $(a, c) \in D \setminus Nil(D)$. Hence, $a \neq 0$. Thus, $(0, k) = (a, c)(0, k/a)$. Hence, $Nil(D)$ is divided in D. Thus, $D \in \mathcal{H}$. Now, it is easy to see that $D/Nil(D)$ is ring-isomorphic to R. Since R is Noetherian domain, we conclude that $D/Nil(D)$ is a Noetherian domain. Hence, D is a Nonnil-Noetherian ring by Theorem 2.2. □

In the following result, we show that for each $n \geq 2$, there is a Nonnil-Noetherian ring with Krull dimension n which is not a Noetherian ring.

Theorem 3.5. *Let R be a Noetherian domain with quotient field K and Krull dimension $n \geq 2$. Then $D = R(+)K \in \mathcal{H}$ is a Nonnil-Noetherian ring with Krull dimension n which is not a Noetherian ring. In particular, if K is the quotient field of $R = Z[x_1, x_2, \ldots, x_{n-1}]$, then $R(+)K$ is a Nonnil-Noetherian ring with Krull dimension n which is not a Noetherian ring.*

Proof. First, by Lemma 3.2 and Proposition 3.3, $Nil(D)$ is not a finitely generated ideal of D. Hence, D is not a Noetherian ring. Now, use an argument similar to that one just given in the proof of Theorem 3.4 to complete the proof of this result. □

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